

# Static and dynamic phase transition in multidimensional voting models on continua

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## Abstract

A voting model (or a generalization of the Glauber model at zero temperature) on a multidimensional lattice is defined as a system composed of a lattice each site of which is either empty or occupied by a single particle. The reactions of the system are such that two adjacent sites, one empty the other occupied, may evolve to a state where both of these sites are either empty or occupied. The continuum version of this model in a  $D$ -dimensional region with boundary is studied, and two general behaviors of such systems are investigated. The stationary behavior of the system, and the dominant way of the relaxation of the system toward its stationary state. Based on the first behavior, the static phase transition (discontinuous changes in the stationary profiles of the system) is studied. Based on the second behavior, the dynamical phase transition (discontinuous changes in the relaxation-times of the system) is studied. It is shown that the static phase transition is induced by the bulk reactions only, while the dynamical phase transition is a result of both bulk reactions and boundary conditions.

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# 1 Introduction

The study of the reaction-diffusion systems, has an attractive area. A reaction-diffusion system consists of a collection of particles (of one or several species) moving and interacting with each other with specific probabilities (or rates in the case of continuous time variable). In the so called exclusion processes, any site of the lattice the particles move on, is either vacant or occupied by one particle. The aim of studying such systems, is of course to calculate the time evolution of such systems. But to find the complete time evolution of a reaction-diffusion system, is generally a very difficult (if not impossible) task.

Various methods have been used to study the reaction-diffusion system: analytical techniques, approximation methods, and simulation. The success of the approximation methods, may be different in different dimensions, as for example the mean field techniques, working good for high dimensions, generally do not give correct results for low dimensional systems. A large fraction of analytical studies, belong to low-dimensional (specially one-dimensional) systems, as solving low-dimensional systems should in principle be easier [1–12].

Various classes of reaction-diffusion systems are called exactly-solvable, in different senses. In [13–15], integrability means that the  $N$ -particle conditional probabilities' S-matrix is factorized into a product of 2-particle S-matrices. This is related to the fact that for systems solvable in this sense, there are a large number of conserved quantities. In [16–24], solvability means closedness of the evolution equation of the empty intervals (or their generalization).

Consider a reaction-diffusion system (on a lattice) with open boundaries. By open boundaries, it is meant that in addition to the reactions in the bulk of the lattice, particles at the boundaries do react with some external source. A question is to find the possible phase transitions of the system. By phase transition, it is meant a discontinuity in some behavior of the system with respect to its parameters. Such discontinuities, may arise in two general categories: in the stationary (large time) profiles of the system, and in the time constants determining the evolution of the system. In the first case, static phase transitions are dealt with; in the second case, dynamical phase transitions. For a review on dynamical phase transitions, one can see for example [25].

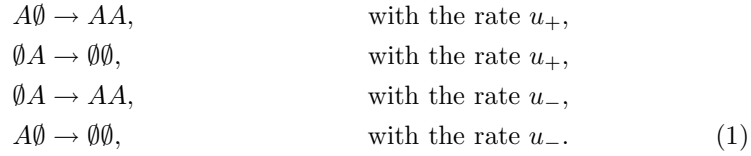
There are systems for them the equation of motion of the one-point function (the probability that a certain site be occupied) is closed, that is independent of the more-point functions [26–28]. Among these systems is the so called voting model (or a generalization of the Glauber model at zero temperature). In [29] a voting system on a one-dimensional lattice was studied, for which at the boundaries of the lattice there are injection or extraction of the particles. Based on the evolution of the one-point functions, it was shown there that the system exhibits two kinds of phase transitions: a static phase transition, corresponding to a discontinuous change in the stationary profile of the one-point function; and a dynamical one, corresponding to a discontinuous change in the behavior of the relaxation time of the system toward its stationary state. In [30–34], the phase structures of extensions of such systems on a one-dimensional lattice were investigated. All of these are restricted to the case of a one-dimensional lattice.

Extending these investigations to higher-dimensional cases would be interesting. Here, we want to study a multi-dimensional extension of the voting model, on a continuum rather than a lattice.

The scheme of the present article is as follows. In section 2, the multidimensional voting model on continuum is presented, and the evolution equation for the density of the particles is obtained. In section 3, the time-independent of the system is studied and it is shown that system exhibits a static phase transition. In section 4, the relaxation of the system toward its stationary state is studied and it is shown that the system exhibits a dynamical phase transition. Section 5 is devoted to the concluding remarks.

## 2 Multidimensional voting models on continua

In [11] and [29], a one dimensional voting model (or a generalization of the Glauber model at zero temperature) on a lattice was defined as follows. Let the system consist of a one-dimensional lattice, each of the sites of which are either empty ( $\emptyset$ ) or containing a single particle ( $A$ ), and let there be a reaction between two neighboring sites like

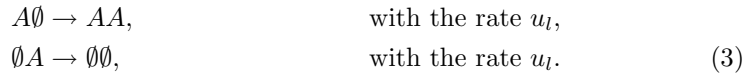


In [11], an open lattice was investigated while in [29], a lattice was studied at the boundaries of which injection and extraction of particles could occur. It was shown that these models are autonomous, meaning that the evolution equation of the  $n$ -point functions contain only  $n$ - or less-point functions. In fact, as it was seen in [29],

$$\frac{d}{dt} \langle n_i \rangle = u_+ \langle n_{i-1} \rangle + u_- \langle n_{i+1} \rangle - (u_+ + u_-) \langle n_i \rangle. \quad (2)$$

Here  $n_i$  is the particle number operator at the site  $i$  of the lattice.

Now consider a multi-dimensional lattice, each site of which is either empty or occupied by a single particle, and let there be a reaction like



Here, we are considering the reaction between a site  $i$  (the right site), which is the ending point of the link  $l$ , and another site (the left site) which is the starting point of the same link. In a one-dimensional lattice, each site is the ending point of two links, which had been denoted by  $+$  and  $-$ . From (3), it is seen that the evolution equation for the one-point function is

$$\frac{d}{dt} \langle n_i \rangle = \sum_l [u_l (\langle n_{i-l} \rangle - \langle n_{i-l} n_i \rangle) - u_l (\langle n_i \rangle - \langle n_{i-l} n_i \rangle)], \quad (4)$$

where by the site index  $i - l$ , it is meant a site which is the starting point of the link  $l$ , the ending point of which is the site  $i$ . It is seen that the two-point function in the right-hand side of (4) cancel each other. So,

$$\frac{d}{dt} \langle n_i \rangle = \sum_l u_l (\langle n_{i-l} \rangle - \langle n_i \rangle). \quad (5)$$

Now, assume that the one-point function is a slowly-varying function of its argument ( $i$ ). In this case, one can define a smooth particle density function of the continuous position variable  $\mathbf{r}$ , with

$$\rho(\mathbf{r}_i) := \frac{1}{\mathcal{V}} \langle n_i \rangle, \quad (6)$$

where  $\mathbf{r}_i$  is position of the lattice site  $i$ , and  $\mathcal{V}$  is the *specific hypervolume* of a site. Then (5) can be rewritten as

$$\frac{\partial}{\partial t} \rho = \sum_l u_l \left[ -\boldsymbol{\delta}_l \cdot \nabla + \frac{1}{2} (\boldsymbol{\delta}_l \cdot \nabla)^2 \right] \rho, \quad (7)$$

where  $\boldsymbol{\delta}_l$  is the link-vector, equal to the position of the ending point of the link  $l$  minus the position of the starting point of the link  $l$ , and higher-derivative terms have been neglected. Using suitable coordinates for  $\mathbf{r}$ , one can write the second derivative as

$$\frac{1}{2} \sum_l u_l (\boldsymbol{\delta}_l \cdot \nabla)^2 = \sum_a \left( \frac{\partial}{\partial x^a} \right)^2, \quad (8)$$

where  $x^a$ 's are the coordinates of  $\mathbf{r}$ . So, (7) is rewritten as

$$\frac{\partial}{\partial t} \rho = (-\mathbf{v} \cdot \nabla + \nabla^2) \rho, \quad (9)$$

where

$$\mathbf{v} := \sum_l u_l \boldsymbol{\delta}_l. \quad (10)$$

Eq. (9) is nothing but a diffusion equation combined with a drift velocity  $\mathbf{v}$ .

Suppose that (9) holds for the interior of the region  $V$ . Integrating (9) on  $V$ , one arrives at

$$\frac{d}{dt} \int_V dV \rho = - \oint_{\partial V} dS \mathbf{n} \cdot \mathbf{v} \rho + \oint_{\partial V} dS \mathbf{n} \cdot \nabla \rho. \quad (11)$$

The first term in the right-hand side is the rate of change of the total number of the particles inside, as a consequence of the drift, while the second term is the effect of injecting or extracting particle at the boundary. The boundary condition

$$\mathbf{n} \cdot \nabla \rho = \alpha - \beta \rho, \quad \text{at the boundary} \quad (12)$$

corresponds to an injection rate of  $\alpha$  per unit hyperarea of the boundary, and an extraction rate of  $\beta$  per unit hyperarea per particle density at the boundary. In general, one can take  $\alpha$  and  $\beta$  position-dependent.

Comparing (9) and (12) with eq. (7) of [29], it is seen that one can transform eq. (7) of [29] to (9) and (12) through

$$\begin{aligned}\delta &= \sqrt{\frac{2}{u+v}}, \\ \mathbf{v} \cdot \hat{\mathbf{x}} &= \delta(u-v), \\ \alpha^- &= \frac{a}{\delta u \mathcal{V}}, \\ \beta^- &= \frac{a+a'}{\delta u}, \\ \alpha^+ &= \frac{b}{\delta v \mathcal{V}}, \\ \beta^+ &= \frac{b+b'}{\delta v},\end{aligned}\tag{13}$$

where the right-hand sides are the quantities defined in [29], and the superscripts  $-$  and  $+$  refer to the left- and right-boundaries, respectively.

From now on, for simplicity we restrict ourselves to the case that the volume  $V$  is a  $D$ -dimensional hyperball with radius  $R$ , the boundary of which is a hypersphere.

### 3 The time-independent state and the static phase transition

Let  $\rho_0$  be the time-independent solution to (9) and (12). Using the ansatz

$$F_{\mathbf{q}}(\mathbf{r}) := \exp(\mathbf{q} \cdot \mathbf{r})\tag{14}$$

(with  $\mathbf{q}$  a constant vector) as a time-independent solution to 9, one arrives at

$$\mathbf{q} \cdot \mathbf{q} - \mathbf{v} \cdot \mathbf{q} = 0,\tag{15}$$

which leads to

$$\mathbf{q} = \frac{1}{2}(\mathbf{v} + \mathbf{v}'),\tag{16}$$

where  $\mathbf{v}'$  is an arbitrary constant vector subject to the condition

$$\mathbf{v}' \cdot \mathbf{v}' = \mathbf{v} \cdot \mathbf{v}.\tag{17}$$

So, one can write the general time-independent solution to (9) as

$$\begin{aligned}
\rho_0(\mathbf{r}) &= \int d\Omega' \tilde{A}(\Omega') F_{\mathbf{q}}(\mathbf{r}), \\
&= \int d\Omega' A(\Omega') \exp \left\{ \frac{1}{2} [(\mathbf{v} + \mathbf{v}') \cdot \mathbf{r} - |\mathbf{v} + \mathbf{v}'| R] \right\}, \\
&=: \int d\Omega' A(\Omega') \exp[G(\mathbf{v}', \mathbf{r})],
\end{aligned} \tag{18}$$

where  $\Omega'$  denotes the angular coordinates of  $\mathbf{v}'$ , and  $A$  is an arbitrary function. It is easy to see that the maximum value of  $G$  is zero, and this maximum value is reached at a point on the boundary ( $r = R$ ), where  $\mathbf{r}$  is parallel with  $\mathbf{v} + \mathbf{v}'$ .

For large values of  $R$  and  $r$ ,  $G$  is a rapidly-varying function and the integral is mainly determined from that point of the integration region which maximizes  $G$ . Generally, there may be two such points: One point is

$$\mathbf{v}'_1 = -\mathbf{v}. \tag{19}$$

The other point is

$$(\mathbf{v}'_2 + \mathbf{v}) \cdot \mathbf{r} = |\mathbf{v}'_2 + \mathbf{v}| r, \quad \text{for } r = R, \tag{20}$$

which means that  $\mathbf{q}$  is parallel with  $\mathbf{r}$ . As the angle between  $\mathbf{q}$  and  $\mathbf{v}$  cannot exceed  $\pi/2$ , the second point exists only if the angle between  $\mathbf{r}$  and  $\mathbf{v}$  is less than  $\pi/2$ . One has

$$G(\mathbf{v}'_1, \mathbf{r}) = 0, \tag{21}$$

and

$$\begin{aligned}
G(\mathbf{v}'_2(\mathbf{r}), \mathbf{r}) &= G[\mathbf{v}'_2(R\mathbf{r}/r), \mathbf{r}] + O[(R-r)^2], \\
&= \frac{(r-R)|\mathbf{v} + \mathbf{v}'_2|}{2} + O[(r-R)^2], \\
&= \frac{(r-R)\mathbf{v} \cdot \mathbf{r}}{R} + O[(r-R)^2].
\end{aligned} \tag{22}$$

Using (21) and (22), one arrives at

$$\rho_0(\mathbf{r}) \sim \begin{cases} C_1(\Omega), & r \sim R, \mathbf{r} \cdot \mathbf{v} < 0, \\ C_1(\Omega) + C_2(\Omega) \exp[\frac{(r-R)\mathbf{v} \cdot \mathbf{r}}{R}], & r \sim R, \mathbf{r} \cdot \mathbf{v} > 0, \end{cases} \tag{23}$$

From this,

$$\nabla \rho_0(r = R) \propto \mathbf{n}(\mathbf{n} \cdot \mathbf{v}) \theta(\mathbf{n} \cdot \mathbf{v}), \quad R \rightarrow \infty, \tag{24}$$

Where  $\theta$  is the step function. It is seen that in the thermodynamic limit ( $R \rightarrow \infty$ ), the density profile at the boundary is stationary, unless  $\mathbf{v} \cdot \mathbf{r} > 0$ . So, changing  $\mathbf{v}$  one can induce a discontinuous change in the slope of the profile density at the boundary. This is the static phase transition, which is seen to be independent of the injection and extraction terms, but dependent on the drift velocity.

## 4 The relaxation of the system toward the stationary state, and the dynamic phase transition

Starting from (9) and (12), one arrives at

$$\begin{aligned}\frac{\partial}{\partial t}(\rho - \rho_0) &= (-\mathbf{v} \cdot \nabla + \nabla^2)(\rho - \rho_0), \\ &=: h(\rho - \rho_0),\end{aligned}\tag{25}$$

and

$$\mathbf{n} \cdot \nabla(\rho - \rho_0) = -\beta(\rho - \rho_0), \quad \text{at the boundary}\tag{26}$$

where  $\rho_0$  is the time-independent solution to (9) and (12). Let  $\psi$  be an eigenfunction of  $h$  corresponding to the eigenvalue  $E$ . Using the ansatz (14) in the eigenvalue equation corresponding to  $h$ , one arrives at

$$\mathbf{q} \cdot \mathbf{q} - \mathbf{v} \cdot \mathbf{q} = E,\tag{27}$$

which leads to

$$\mathbf{q} = \frac{1}{2}(\mathbf{v} + \mathbf{v}'),\tag{28}$$

where  $\mathbf{v}'$  is an arbitrary constant vector subject to the condition

$$\mathbf{v}' \cdot \mathbf{v}' = \mathbf{v} \cdot \mathbf{v} + 4E.\tag{29}$$

So, one has

$$\psi(\mathbf{r}) = \exp(\mathbf{v} \cdot \mathbf{r}/2) \int d\Omega' A(\Omega') \exp(\mathbf{v}' \cdot \mathbf{r}/2),\tag{30}$$

where  $A$  is to be found so that the boundary condition (26) is satisfied with  $\psi$ .

If the righthand side of (29) is positive, then  $\mathbf{v}'$  is real and for large  $r$ , one can approximate  $\psi$  like

$$\psi(\mathbf{r}) \sim \exp(\mathbf{v} \cdot \mathbf{r}/2) A(\Omega) \exp(v' r/2),\tag{31}$$

where  $\Omega$  is the angular coordinates corresponding to  $\mathbf{r}$ . The boundary condition (16), then becomes

$$\left[ \frac{v'}{2} + \beta(\Omega) + \frac{\mathbf{n} \cdot \mathbf{v}}{2} \right] A(\Omega) = 0.\tag{32}$$

This has a nonzero solution for  $A$ , provided the parenthesis vanishes for some  $\Omega$ . As  $\beta \geq 0$ , this happens for some (real) positive  $v'$ , if and only if

$$\min \left[ \beta(\Omega) + \frac{v \cos \phi}{2} \right] < 0,\tag{33}$$

where  $\phi$  is the angle between  $\mathbf{r}$  and  $\mathbf{v}$ . If (33) holds, then the range of  $v'$  for which a nonzero solution to (32) for  $A$  exists is

$$0 \leq v' \leq -\min \left[ \beta(\Omega) + \frac{v \cos \phi}{2} \right]. \quad (34)$$

(This is true for more-than-one dimensional space. If the space is one-dimensional,  $v'$  has only one acceptable value, as the parenthesis in (32) has only two values at most one of them can be zero.)

If (33) holds, then there exists eigenvalues  $E$  for  $h$ , with  $E > -\mathbf{v} \cdot \mathbf{v}/4$ . Otherwise, all of the eigenvalues of  $h$  are less than or equal to  $-\mathbf{v} \cdot \mathbf{v}/4$ . The relaxation time of the system is

$$\tau = -\frac{1}{E_{\max}}, \quad (35)$$

where  $E_{\max}$  is the largest eigenvalue of  $h$ . The largest value of  $E$  is either  $-\mathbf{v} \cdot \mathbf{v}/4$ , or the value obtained from (29) for the largest value of  $v'$ . So,

$$\tau = \begin{cases} \frac{4}{\mathbf{v} \cdot \mathbf{v}}, & \min \left[ \beta(\Omega) + \frac{v \cos \phi}{2} \right] > 0 \\ \frac{4}{\mathbf{v} \cdot \mathbf{v} - \{\min[2\beta(\Omega) + v \cos \phi]\}^2}, & \min \left[ \beta(\Omega) + \frac{v \cos \phi}{2} \right] < 0 \end{cases} \quad (36)$$

In the first case, the system is in the fast dynamical phase, in which the relaxation time does not depend on the boundary condition. In the second case, the system is in the slow dynamical phase, in which the relaxation time is larger and does depend on the boundary conditions. This is the dynamical phase transition.

## 5 Concluding remarks

It was seen that the so-called voting model defined on a one-dimensional lattice, has a natural analog on a multidimensional continuum. It was seen that there are two kinds of phase transition, a static one corresponding to a discontinuous change in the behavior of the stationary profile of the system, and a dynamical phase transition corresponding to the relaxation of the system toward its stationary state. The static phase transition is controlled by the bulk reactions, while the dynamical phase transition is controlled by the bulk reactions and the boundary conditions both. This is analogous to what seen for the case of a one-dimensional lattice.

There are, however, differences. In the multidimensional case, the static transition occurs when the direction of the drift velocity is changed. This can happen without being necessary that the drift velocity vanishes. In the one-dimensional case, however, the static phase transition occurs only when the drift velocity passes zero. The reason is that in one-dimension, the only way to change the direction of a vector smoothly, is that the vector vanishes at some point.



The second difference concerns the dynamical phase transition, to be more precise, the largest eigenvalues of the operator  $h$  defined in (25). In the one-dimensional case and in the slow phase, there is only one eigenvalue greater than the largest eigenvalue corresponding to the fast phase. In the multidimensional case, however, in the slow phase the spectrum of  $h$  contains a continuous region the lower bound of which is the largest eigenvalue of  $h$  in the fast phase. This means that in the one-dimensional case and in the slow phase, there is a largest relaxation time and a gap between this and the next largest relaxation time, while in the multidimensional case, there is no such gap.

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